[4] D A Hill, "Reflection coefficient of a waveguide with slightly uneven walls," *IEEE Trans Microwave Theory Tech*, vol 37, pp 244–252, Jan 1989
[5] A. K Mallick and G S. Sanyal, "Electromagnetic wave propagation in a rectangular waveguide with sinusoidally varying width," *IEEE Trans Microwave Theory Tech*, vol MTT-26, pp. 243–249, Apr. 1978
[6] J A Kong, *Electromagnetic Wave Theory* New York. Wiley, 1986.
[7] M. J. Kelly, "Thermal anomalies in very fine structures," *J Phys. C*, vol 15, pp L969, 1982
[8] P M Morse and H Feshbach, *Methods of Theoretical Physics*. New York: McGraw-Hill, 1953, pp 1071
[9] W A Gardner, *Introduction to Random Processes* New York Macmillan, 1986, pp 57
[10] Y A Kravtsov and A. I Saichev, "Effects of double passage of waves in randomly inhomogeneous media," *Sov. Phys—Usp* vol. 25, pp. 494–508, July 1982

Spectral Estimation for the Transmission Line Matrix Method

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***Abstract*—Spectral estimation for the transmission line matrix (TLM) method by use of the discrete Fourier transform and fast Fourier transform is reviewed. Error bounds are given and checked by means of a numerical example. A new spectral estimation method based on Prony's method is presented for use with TLM. A numerical example shows that the new method allows an order of magnitude reduction in the number of iterations in the TLM method for equal accuracy.**

I. Introduction

The transmission line matrix (TLM) method for microwave circuit analysis calculates the time-domain variation of the electromagnetic fields in response to an arbitrarily chosen excitation [1]. Because of the discrete nature of the TLM method the output waveform is not a continuous function; it is a sequence of delta functions of varying amplitude. These delta functions are separated in time by Δt, which depends on the cell size used in the TLM model. This time is given by

$$\Delta t = \Delta l / c$$

where Δl is the spacing between nodes and c is the velocity of light in free space.

Frequently the desired information from TLM analysis is not the time-domain response but rather frequency-domain information. A common use of TLM is to determine the resonant frequencies of the characteristic modes of a microwave structure. To obtain these frequency-domain data we must apply some spectral estimation method to the time-domain output data. In the remainder of this paper we shall review spectral estimation methods presently in use for TLM, suggest an alternative spectral estimation method which appears promising, and compare numerical results for a typical problem.

II. Present Methods

In the original paper describing TLM, Johns and Beurle used Fourier transform techniques to obtain the frequency response of the circuit [2]. Specifically they applied the Fourier integral to the sequence of delta functions which represented the time-domain response of the circuit. By an application of the sifting property of the delta function, they expressed the frequency-domain response as a pair of finite summations:

$$\mathrm{Re}\left(F\left(\frac{\Delta}{\lambda}\right)\right) = \sum_{k=1}^{NI} I_k \cos\left(2\pi k \frac{\Delta l}{\lambda}\right) \quad (1)$$

$$\mathrm{Im}\left(F\left(\frac{\Delta l}{\lambda}\right)\right) = \sum_{k=1}^{NI} I_k \sin\left(2\pi k \frac{\Delta l}{\lambda}\right) \quad (2)$$

where $F(\Delta l/\lambda)$ is the frequency response, I_k is the output impulse response at time $t = k(\Delta l/c)$, and NI is the total number of iterations used in the TLM method.

We can then form either $|F(\Delta l/\lambda)|$ or $|F(\Delta l/\lambda)|^2$ (which are analogous to voltage magnitude or power respectively) and plot this as a function of frequency. Resonant frequencies of the microwave circuit correspond to peaks (local maxima) of the plot.

These equations are written in a normalized form. Johns and Beurle rescaled so as to make the interval between pulses unity rather than Δt. Thus they divided all times by Δt and multiplied all frequencies by Δt. We note that

$$\Delta t f = \frac{\Delta l}{c} f = \frac{\Delta l}{\lambda f} f = \frac{\Delta l}{\lambda} \quad (3)$$

which gives the frequency variable used in (1) and (2). By Nyquist's criterion, $f \leqslant 1/2\Delta t$, which ensures that $0 \leqslant \Delta l/\lambda \leqslant 0.5$.

There are two sources of error in determining resonant frequency in this manner. The first is a truncation error. One can only run the TLM simulation for a finite number of iterations, which we denote by NI. This has the effect of viewing the true (infinitely long) time-domain response through a rectangular window. The duration of this window is $NI\Delta t$. In the frequency domain, the effect of this windowing is to convolve the true frequency spectrum with the function

$$F_{\mathrm{window}}\left(\frac{\Delta l}{\lambda}\right) = \frac{\sin\left(\pi NI \Delta t \frac{\Delta l}{\lambda}\right)}{\pi NI \Delta t \frac{\Delta l}{\lambda}}. \quad (4)$$

The effects of this convolution are twofold. It widens the peaks in the frequency response plot, and the side lobes of the $\sin x/x$ function cause the side lobes due to one response peak to overlap the main lobe of another response peak. The result is that the observed local maxima of the frequency response are shifted away from their true values. Johns has derived an error bound for this shift [3]. For two response peaks of equal amplitude, separated by normalized frequency $S = \Delta l/\lambda$, Johns found the maximum truncation error $\Delta S_{\mathrm{trunc}}$ to be bounded by

$$\Delta S_{\mathrm{trunc}} \leqslant \pm \frac{3}{S(NI\pi)^2}. \quad (5)$$

An additional error source arises in finding the peaks in the response curve. While the frequency response given by the finite summation in (5) is a continuous function of frequency, we can only evaluate this equation at a finite number of points. If we evaluate the frequency response at NF points equally spaced across the interval (0, 0.5) any peak we find may be shifted from its true position by a normalized frequency of $\pm 1/4NF$. This

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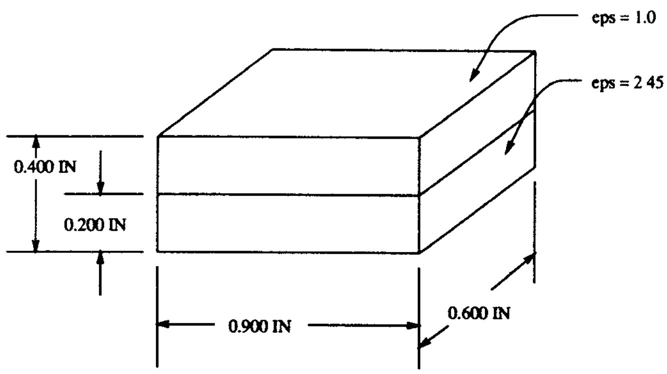

Fig. 1 Geometry of the waveguide resonator.

suggests that we choose NF so that

$$NF \geqslant \frac{1}{4\Delta S_{\text{peak}}} \tag{6}$$

where ΔS_{peak} is the maximum acceptable normalized frequency error. We should note that (6) is a conservative bound; use of interpolation or curve-fitting techniques can provide improved accuracy.

Evaluation of the frequency response may be done one frequency at a time by the discrete Fourier transform (DFT). This was the method originally used by Johns [4]. If it is desired to find the frequency response at a number of equally spaced frequencies, then it is computationally efficient to replace the DFT with the fast Fourier transform (FFT). The FFT is now widely used in TLM calculations [1].

It should be emphasized that using the FFT does not require that $NI = NF$. If accuracy considerations based on (5) and (6) suggest that $NF > NI$, then the NI impulses from the TLM simulation can be extended by zero-padding the data to provide NF samples for the FFT routine. While this does not alter the width of the $\sin x / x$ spreading, it does provide closer spaced samples in the frequency domain, which improves the accuracy of the peak finding [5].

III. Numerical Results

We present numerical results for an inhomogeneous cavity resonator. The resonator consists of WR-90 rectangular waveguide half filled with air ($\epsilon_r = 1.0$) and half filled with polystyrene ($\epsilon_r = 2.45$). The length of the waveguide is 0.600 in and a conducting wall covers each end of the waveguide. Fig. 1 illustrates the geometry of the resonator. This geometry was chosen because it is an inhomogeneous problem that can be solved analytically, which provides an independent check of the numerical results [6].

A symmetrical condensed node TLM algorithm was used to determine the dominant resonant frequency. This algorithm is due to Johns [7]. Cubical nodes were used, with a nodal spacing of 0.100 in, yielding a nodal array of dimensions $9 \times 4 \times 6$. The E_y field was excited in the dielectric-filled region in the center of the resonator. The output field sampled was the E_y field in the center of the cavity (node 4, 2, 3). The number of iterations NI ranged from 128 to 4096 in powers of 2. The number of frequencies evaluated, NF, also ranged from 128 to 4096, independent of NI.

The relatively coarse mesh chosen causes appreciable modeling error. This error was estimated by first finding the exact resonant frequency by separation of variables, and then running the TLM method for an extremely large number of iterations (which is normally impractical due to the excessive computer time required) so that the truncation error becomes insignificant.

TABLE I
Transmission Line Matrix with Fast Fourier Transform

NI	Iteration Time (sec)	NF	FFT Time (sec)	Frequency (GHz)	Fractional Error
128	158.9	128	1.9	9.22108	-1.01%
		256	2.9	9.68213	+3.94%
		512	3.7	9.68213	+3.94%
		1024	6.3	9.56687	+2.71%
		2048	11.6	9.56687	+2.71%
		4096	23.2	9.56687	+2.71%
256	313.0	256	2.6	9.22108	-1.01%
		512	3.8	9.45160	+1.47%
		1024	6.3	9.33634	+0.23%
		2048	11.7	9.39397	+0.85%
		4096	23.3	9.39397	+0.85%
512	621.0	512	3.7	9.22108	-1.01%
		1024	6.4	9.33634	+0.23%
		2048	11.8	9.33634	+0.23%
		4096	23.5	9.33634	+0.23%
1024	1237.7	1024	6.4	9.33634	+0.23%
		2048	11.8	9.33634	+0.23%
		4096	23.6	9.30752	-0.08%
2048	2472.0	2048	12.2	9.33634	+0.23%
		4096	23.8	9.30752	-0.08%
4096	4937.0	4096	24.2	9.30752	-0.08%

The TLM method run with $NI = 16384$ and $NF = 16384$ gives the dominant resonance at 9.315 GHz and the next higher resonance at 21.612 GHz. Analytic results give the dominant resonance (corresponding to the LSM_{10} waveguide mode) at 9.362 GHz, with the next resonance occurring at 20.910 GHz. The modeling error is -0.49 percent for the lowest resonance and $+3.35$ percent for the higher resonance. These errors are a function of the spacing between the nodes. The chosen spacing of 0.100 in corresponds to 12.60 cells per wavelength at 9.362 GHz and 5.64 cells per wavelength at 20.910 GHz. By spacing the nodes more closely we could reduce the modeling error. This was not done in order to minimize the computational effort involved in running these test cases.

For the remainder of this paper we shall ignore the modeling error and concentrate on the error in frequency estimation. In Table I we list the calculated resonant frequencies for the various values of NI and NF, as well as the fractional error relative to the converged TLM result. If we desire an accuracy better than ± 0.10 percent, we see that we must choose $NI \geqslant 1024$ and $NF \geqslant 4096$. For the case $NI = 1024$, $NF = 4096$ the TLM iterations take 1238 s. The frequency estimation using the FFT takes 24 s. If frequency estimation is done by a DFT instead of the FFT then the frequency estimation takes 3267 s. These timings were obtained using an 80286/80287 based personal computer clocked at 10 MHz.

Let us compare these results with the error bounds given in Section II. By applying (5) to the dominant and next higher resonance (and their reflections in the negative frequency part of the spectrum) we find that $NI = 512$ gives an error smaller than ± 0.114 percent and $NI = 1024$ gives an error smaller than ± 0.028 percent. Application of (6) to the dominant resonance indicates that we need $NF \geqslant 6329$ to ensure an error less than ± 0.10 percent. These bounds show reasonable agreement with the numerical results obtained.

IV. Prony's Method

Why does the FFT lead to errors in frequency estimation? The time-domain impulse response of a lossless microwave circuit at a fixed observation point can be expressed as a summation over the resonant modes:

$$G(t) = \sum_{n=1}^{\infty} a_n \cos 2\pi f_n t + b_n \sin 2\pi f_n t \tag{7}$$

where a_n and b_n are modal amplitudes and f_n are the resonant frequencies. When we approximate this impulse response using the FFT we obtain a summation of the form

$$G(t) = \sum_{n=1}^{\infty} a_n \cos 2\pi n f_N t + b_n \sin 2\pi n f_N t \tag{8}$$

where $f_N = 1/(N\Delta t)$. The frequencies used in the FFT-derived estimate are all harmonics of f_N, which is the fundamental frequency associated with the observation period. But the actual resonant frequencies are not harmonically related, so a mismatch between the actual resonant frequencies and the FFT frequencies is unavoidable. This mismatch is the source of both the truncation and peak finding errors.

To avoid these errors we could try to fit an expansion such as (7) directly to the TLM data by choosing a_n, b_n, and f_n appropriately. This can overcome the frequency mismatch problem. It is not possible to let the summation range from 0 to ∞ so we must truncate the summation at some upper limit NF. Since in the true impulse response, the coefficients a_n and b_n tend to 0 as $n \to \infty$, the effects of this truncation can be made as small as desired by choosing NF large enough. We are then left with the problem of choosing a_n, b_n, and f_n so that

$$G(t) = \sum_{n=1}^{NF} a_n \cos 2\pi f_n t + b_n \sin 2\pi f_n t \tag{9}$$

gives the best possible fit to the data resulting from the TLM simulation.

A conventional application of least-squares techniques to (9) gives a set of nonlinear equations which are computationally impractical. Baron de Prony found an ingenious method for solving this class of problem in 1795. His method replaces the nonlinear set of equations with a single polynomial. The roots of this polynomial determine the frequencies f_n. A detailed explanation of the method is given by Hildebrand [8]. We provide an outline of the method as follows:

We define a rectangular matrix $\boldsymbol{C}$ having dimension $(ND - 2NF)$ by (NF) where

$$c_{ij} = \begin{cases} I_{i+j} + I_{i-j+2NF}, & j \neq NF \\ I_{i+NF}, & j = NF \end{cases} \tag{10}$$

and a vector $\vec{d}$ of length $(ND - 2NF)$ by

$$d_i = -I_i - I_{i+2NF}. \tag{11}$$

We then find a vector $\vec{x}$ which solves the equation

$$\boldsymbol{C} \cdot \vec{x} = \vec{d} \tag{12}$$

either exactly or in a least-squares sense. The vector $\vec{x}$ determines a polynomial $P(\omega)$ which may be expressed as

$$T_{NF+1}(\cos\omega) + x_1 T_{NF}(\cos\omega) + \cdots + x_{NF-1} T_1 + \frac{1}{2} x_{NF} T_0 = 0 \tag{13}$$

where the T_n are the Chebyshev polynomials. We then find the roots of the polynomial and take the inverse cosine to obtain $\omega_1, \cdots, \omega_{NF+1}$, which are the desired angular frequencies. The f_n are given by

$$f_n = \frac{\omega_n}{2\pi}. \tag{14}$$

Now that we have determined the frequencies, the modal amplitudes a_n and b_n can be found by solving a second linear least-squares problem.

Prony's method has been previously used in electromagnetics [9]. When it is used with experimental data, problems may arise, as the method is sensitive to noise in the data. Also the choice of order is critical, as either too high or too low an order can give poor accuracy, and there is no good criterion for selection of the order [10]. When used with TLM data these problems are alleviated. The TLM simulation is inherently noise-free (except for round-off errors, which are usually insignificant). Selection of the order is also not critical. While use of too low an order gives poor accuracy, use of too large an order gives accurate results at the expense of a greater computation time.

TABLE II
Transmission Line Matrix Using Prony's Method

NI	Iteration Time (sec)	NF	Prony Time (sec)	Frequency (GHz)	FP	Relative Error
32	43.01	15	37.73	9.58274	.9907	+2.877 %
64	81.57	15	78.05	9.32653	.99972	+0.127 %
128	158.68	15	154.12	9.31658	.999173	+0.020 %
256	312.74	15	281.82	9.31484	.998288	+0.000 %
512	621.21	15	557.17	9.31541	.996391	+0.007 %
1024	1237.86	15	1094.83	9.31570	.995713	+0.010 %

V. Numerical Results

We have used Prony's method in association with condensed-node TLM to obtain the resonant frequencies of the same inhomogeneous waveguide resonator that was described in Section III. The linear least-squares problems were solved by a singular value decomposition technique. Laguerre's method was used for the polynomial root finding [11]. To conserve memory the TLM simulation used single precision arithmetic but it was found to be necessary to use double precision arithmetic throughout the subroutine that implemented Prony's method.

Both NI, the number of iterations, and NF, the number of frequencies to fit, were varied. In Table II we list the results obtained for NI ranging from 32 to 1024 in powers of 2. NI was fixed at 15 for these runs. Justification for this choice of NI is given later. To obtain ± 0.10 percent accuracy it is necessary to choose $NI \geqslant 128$. This is a considerable reduction compared with the number of iterations needed to obtain comparable accuracy using a FFT.

The value selected for NF has a significant effect on the performance of Prony's method. If too small a value is chosen, the accuracy of the spectral estimate suffers, as the estimate

TABLE III
TLM USING PRONY'S METHOD ($NI = 128$; NF VARIABLE)

NF	Time Prony (sec)	Frequency (GHz)	FP	Fractional Error
1	1.0	6.54095	0.001	-29.78%
2	3.4	5.41799	0.000	-41.83%
3	7.0	3.30758	0.020	-64.49%
4	11.2	2.77753	0.134	-70.18%
5	17.0	2.40053	0.070	-74.23%
6	25.4	2.18275	0.037	-76.57%
7	37.0	1.85374	0.009	-80.10%
8	44.0	1.09399	0.506	-88.26%
9	50.8	9.64002	0.979	+3.49%
10	62.7	9.40548	0.993	+0.97%
11	79.5	9.29897	0.995	-0.17%
12	93.9	9.30600	0.998	-0.09%
13	109.1	9.29602	0.999	-0.20%
14	124.0	9.31200	0.999	-0.03%
15	155.4	9.31658	0.999	+0.02%
16	168.2	9.31399	0.999	-0.01%
17	190.1	9.30985	0.999	-0.05%
18	212.3	9.31218	0.999	-0.03%
19	245.8	9.31340	0.999	-0.01%
20	263.6	9.31429	0.999	-0.01%

cannot model all the significant frequency components in the TLM simulation. If NF is chosen too big the execution time of the program is increased. Also choosing NF too big requires solving large matrices and rooting polynomials of large order, which may lead to numerical instability. How shall we select NF?

One simple approach is to choose a very small value for NF, perform the calculations, and then increment NF and repeat the calculations. This process is continued while monitoring the frequency values of the spectral components of interest. When the frequency values have converged to the desired accuracy the process is terminated.

A more efficient approach is to measure directly the quality of the spectral estimate obtained using Prony's method. We select a value for NF and use Prony's method to determine both the frequencies f_n and the amplitudes a_n, b_n. Then for a time step k in the TLM simulation the actual response is I_k and the estimated response due to Prony's method is I_k^{nf}, which is given by

$$I_k^{nf} = G(k\Delta t) \tag{15}$$

with $G(t)$ defined by (9). Then we define the residual R_k^{nf} by

$$R_k^{nf} = I_k - I_k^{nf}. \tag{16}$$

We then compare the power in the residual to the power in the TLM simulation. The fractional power FP is given by

$$FP = 1 - \frac{\sum_{k=1}^{NI} \left(R_k^{nf}\right)^2}{\sum_{k=1}^{NI} \left(I_k\right)^2}. \tag{17}$$

The fractional power can be used to test the value of NF. If $FP \geqslant 0.999$ then our experience is that the value for NF is satisfactory. If $FP < 0.999$ then the value of NF should be increased.

Table III gives results for $NI = 128$ and $1 \leqslant NF \leqslant 20$. We see that if $NF \leqslant 8$ Prony's method is grossly inaccurate and that $FP \leqslant 0.60$. If $NF \geqslant 13$ then $FP > 0.999$ and the accuracy is satisfactory. The data also show that the exact choice of NF is not critical provided NF is large enough.

VI. CONCLUSIONS

FFT spectral estimation methods assume that the various frequency components in the TLM simulation are harmonically related. A more accurate spectral estimate is possible by allowing both the frequencies and the amplitudes to vary. These amplitudes and frequencies may be found by using Prony's method. Prony's method allows about a 10 to 1 reduction in the number of steps in the TLM simulation for equal accuracy when compared with FFT methods. Further work is needed to find more efficient ways to implement Prony's method. Improvements may be possible in the polynomial rooting, as we know that the roots are all real roots and lie in the interval $[-1,1]$.

REFERENCES

[1] W. J. R. Hoefer, "The transmission-line matrix method—Theory and application," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 882–893, Oct. 1985.

[2] P B. Johns and R L. Beurle, "Numerical solution of 2-dimensional scattering problems using a transmission-line matrix," *Proc Inst Elec. Eng.*, vol 118, no. 9, pp. 1203–1208, Sept. 1971.

[3] P B Johns, "Application of the transmission-line-matrix method to homogeneous waveguides of arbitrary cross-section," *Proc Inst. Elec. Eng.*, vol 119, no. 8, pp. 1086–1091, Aug. 1972

[4] P. B. Johns, "Solution of inhomogeneous waveguide problems using a transmission-line matrix," *IEEE Trans. Microwave Theory Tech*, vol MTT-22, pp. 209–215, Mar. 1974.

[5] S. M. Kay, *Modern Spectral Estimation* Englewood Cliffs, NJ: Prentice-Hall, 1987, pp. 76–77

[6] R F. Harrington, *Time Harmonic Electromagnetic Fields* New York: McGraw-Hill, 1961, pp 158–161.

[7] P. B Johns, "A symmetrical condensed node for the TLM method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 370–377, Apr. 1987.

[8] F. B. Hildebrand, *Introduction to Numerical Analysis* New York: McGraw-Hill, 1974, pp. 457–465.

[9] M. L. Van Blaricum, "A review of Prony's method techniques for parameter estimation" in *Proc Rome Air Development Center Spectrum Estimation Workshop* (Griffis Air Force Base), May 24–26, 1978, pp. 125–139.

[10] M. L. Van Blaricum and R. Mittra, "Problems and solutions associated with Prony's method for processing transient data," *IEEE Trans. Antennas Propagat*, vol AP-26, pp. 174–182, Jan. 1978

[11] W. H. Press, B. P. Flannery, S A Teukolsky, and W T Vetterling, *Numerical Recipes*. Cambridge· Cambridge University Press. 1986, pp 59–64, 515–520, 263–266